A Gradient-based Method for Analyzing Stochastic Variational Inequalities with One Uncertain Parameter

Department of Operations and Information Management
Working Paper 00-03-13

Margaret H. Belknap
Department of Systems Engineering
University of Pennsylvania
Philadelphia, PA 19104-6315
belknap@seas.upenn.edu

Chun-Hung Chen
Department of Systems Engineering
University of Pennsylvania
Philadelphia, PA 19104-6315
chchen@seas.upenn.edu

Patrick T. Harker
The Wharton School
University of Pennsylvania
Philadelphia, PA 19104-6366
harker@wharton.upenn.edu

A Gradient-based Method for Analyzing Stochastic Variational Inequalities with One Uncertain Parameter

ABSTRACT

We present a method of estimating statistics of interest for a Stochastic Variational Inequality with one uncertain parameter, λ. Our method significantly reduces the heavy computational effort associated with generating a sample space of N solutions for N random values of λ. Starting with a relatively small sample of solutions for values of λ, we use the gradient information at those solution points to determine whether we may estimate solutions or must solve for actual solutions for interim values of λ. We continue in an iterative manner to generate a sample space of size N that consists of estimated solutions for some values of λ and actual solutions for others. Using estimates in lieu of actual solutions represents significant computational savings. We extend our methodology for the specific objective of calculating the probability of whether solution elements take on values above a threshold. In this case, even fewer actual solutions need to be calculated.

1. Introduction

Variational inequalities may be used to model a broad range of problems across a diverse set of disciplines. Cottle, Giannessi and Lions (1980) characterize those who work with variational inequalities into two general categories. In the first category are those interested in physical models, differential equations, and topological vector spaces. Examples in this category include optimal control, free boundary, and obstacle problems. Kinderlehrer and Stampacchia (1980) and Baiocchi and Capelo(1984) are two excellent texts on the mathematics of variational inequalities and applications in this category.

In the second category, labeled by Cottle, Giannessi and Lions (1980) as the "complementarity people," are those primarily interested in areas outside the "classical" areas of applied science (such as operations research) leading to the study of problems in mathematical programming. Perhaps this second category might be more aptly referred to as those interested in the study of general equilibrium. We count ourselves, in the context of this paper, as belonging to this second category. Hence, the breadth of the applicability of our work is as broad as the fields of economics, operations research and management science. After introducing theory, fundamentals and some solution techniques for variational inequalities, Nagurney (1999) devotes the remainder of her text to applications in general equilibrium.

A brief history of variational inequalities is included in Harker (1992). Harker and Pang (1990) detail a complete survey of literature and methods in both categories of research up until 1990 and include an extensive bibliography. Since then, Harker published his lectures on General Equilibrium in 1992. In 1993, Nagurney published a book on Variational Inequalities in Economics that was updated in 1999. Ferris (1999) maintains a website on Mixed Complementarity Problems. It includes an extensive library of problems formulated in GAMS and access to the PATH solver (Ferris and Munson, 1999) that we used to test our methodology. These publications together with available internet resources stand as evidence of the growing significance and wide use of Variational Inequalities, in general, and in economics, transportation, and telecommunications, in particular.

This paper is a presentation of a method of estimating statistics of interest for a Stochastic Variational Inequality (SVI). We begin by providing three definitions:

Definition 1. *The* variational inequality*: Denoted VI (X, F) with X a nonempty subset of* R^n*, F a mapping,* R^n R^n*, the problem is to find the solution vector, x* such that the inner product,* $\langle F(x^*),\ x - x^* \rangle \geq 0$*, is satisfied for all x.*

Definition 2. *The* parametric variational inequality*: Denoted PVI(X,F, λ) with X a nonempty subset of* R^n*,* $\Lambda \subseteq R^p$*,* the space of parameters, $\lambda \in \Lambda$ *,F a mapping,* R^n $x\,\Lambda$ R^n*, the problem is to find the solution vector, x*(λ) such that the inner product* $\langle F(x^*(\lambda)),\ x - x^*(\lambda) \rangle \geq 0$ *is satisfied for all x.*

Definition 3. *The* stochastic variational inequality*: Denoted, SVI(X,F,λ),when λ is a random vector with a suitably defined probability distribution, the problem is to find the distribution of x(λ) where for a given outcome, λ' , x(λ') is a solution of the PVI(X,F, λ').*

As the use of variational inequalities increases, particularly in the aforementioned fields, so too increases the importance of considering the stochastic variational inequality problem. The reason for this should be quite evident. That is, often times such things as price elasticities, demand elasticities, consumer demand, friction coefficients, and traffic densities are uncertain. Yet, analysts frequently substitute the mean or a point estimate for uncertain parameters and solve the problem. This practice is particularly alarming when applied to a variational inequality problem because the true mean of the solution may be quite different than the solution obtained by this method. Moreover, in practice, policy makers need to know more than what the single solution point is. They are interested in such questions as, “What is the probability that X is greater than a critical value?”

This same criticism applies to the practice of using point estimates for a stochastic version of the linear optimization problem (LP). However, most LP solvers provide a fairly comprehensive sensitivity analysis that may be used to shed light on a range of solutions. In sharp contrast, the solution to a Variational Inequality offers only a solution point. Then, under various assumptions of continuity and differentiability (see Nagurney 1999 for essential details, and Murdukhovich 1994 for lengthy discussion and bibliography), the analyst may apply the implicit function theorem to derive gradients at the solution point (see Tobin 1986 for a numerical example). The existence of gradients tells the analyst that solutions exist in a very small neighborhood near the solution point. And the sign of the gradient indicates the direction of change with respect to the parameter of interest. Clearly, for a stochastic variational inequality a sensitivity analysis from a single solution point is insufficient for providing an analysis robust enough to support policy and decision-making.

It is important to distinguish our work on stochastic variational inequalities to estimate statistics of interest from sensitivity analysis. Sensitivity analysis, as defined in the literature (Tobin, 1986) is concerned with the continuity of the solution from a given point. Meanwhile, this analysis is concerned with establishing the answers to such questions as: What are the means of the solution values over the range of the uncertain parameter? What is the probability that a solution element, x_i, is positive or more generally greater than or equal to a specified threshold?

In this paper, we propose a method for generating a large, say N-size, sample space to estimate the mean of a vector element (variable) of interest and to estimate probabilities that a variable takes on a range of values. We do this without resorting to solving all N associated variational inequalities. Instead, we start with a relatively small sample of solutions for values of λ. We calculate the gradient information at those solution points and use them to estimate solutions for interim points. Then, based upon these estimates we determine whether it is necessary to solve for actual solutions for interim values of λ. We continue in an iterative manner to generate a sample space of size N that consists of estimated solutions for some values of λ and actual solutions for others.

Using estimates in lieu of actual solutions represents significant computational savings. While we have not taken on the daunting task of specifying the entire distribution of the solution vector, we have improved upon the usual practice of simply substituting the mean. Further, we have developed a method that is computationally much less burdensome than simulating N variational inequalities and solving them all to generate a sample space. The resulting sample space may be used to estimate the mean, as well as probabilities of interest. We extend our methodology for the specific objective of calculating the probability of whether solution elements take on values above a threshold. In this case, even fewer actual solutions need to be calculated.

The rest of this paper is organized as follows. The next section is a literature review. In Section 3, we present our method. In Section 4, we illustrate our method with a numerical example using two different distributions for the stochastic parameter. We compare our results to those obtained by simply using a point estimate of the mean and to results obtained if we had calculated all solutions rather than estimating some and calculating others. We close with conclusions and areas for further research in Section 5.

2. Literature Review

In spite of its importance, little has been published on stochastic variational inequalities as we have defined them. Gurkan, Ozge and Robinson (1999) determine "average" equations f(x) through simulation and then solve a single variational inequality. They demonstrate that their method is superior to that of simply using the mean and provide an example. However, this approach cannot address the kinds of questions that we pose in the introduction regarding means and probabilities. De Wolf and Smeers (1997) approximate uncertain demand curves with piecewise linear curves in a Stackelberg-Nash-Cournot Equilibrium model and solve an optimization problem. Because they assume a discrete set of M possible scenarios for their demand curves, they can assign probabilities to each of their solutions. In our work the uncertain parameter may have a continuous distribution.

Harrison et al. (1993) and Harrison and Vinod (1992) do not refer to their problem as a SVI. However, their problem, a PVI with stochastic parameters, fits our definition.

Also, they term their work “systematic sensitivity analyses” varying just one parameter and then varying multiple parameters for specific values. Regardless of semantics, their work addresses the questions we seek to answer.

Harrison et al. (1993) estimated the mean, referred to as the point estimate, PE, and standard error, SE, for each stochastic parameter in their PVI. When varying one parameter, they perform simulations on 5 points. The points are the PE, the PE +/- 0.7 SE, and PE+/-1.4 SE. From these 5 points, they calculate probabilities that vector elements are above a threshold value. They cite computational burden as the reason that they do not perform more runs. We believe that this is where our work will make a significant contribution because we are able to generate a large sample space from relatively few points. Moreover, for their particular objective, we could determine from the calculation of two points, e.g. the PE and the PE-1.4SE, whether or not the interim point (PE-.0.7SE) needs to be calculated.

When varying more than one parameter, say m=3, Harrison et al. (1993) simulate all possible combinations of these five values, K=5 (i.e. 125). Then, assigning appropriate weights to these solutions, they again calculate probabilities of interest. In Harrison and Vinod (1992), the authors first create a possible set of discrete combinations, K^m, over continuous values of stochastic parameters and use this pool to create a sample (with replacement) of a subset of combinations for which they obtain solutions. More specifically, they allow 48 parameters, K, to take on m=4 distinct and equiprobable values. Then they sample 15,000 combinations and solve 15,000 variational inequalities. From these 15,000 solutions, they can estimate a mean solution and calculate confidence intervals. Generating sample spaces when more than one parameter is stochastic is an area that we look forward to researching further. We hope to extend this work in a subsequent paper to accommodate multiple stochastic parameters.

3. The Proposed Methodology

Since the solution to SVI(F(x), λ) cannot generally be represented in closed-form, we must rely on simulation to obtain relevant insights to the nature of the distribution of its solution, X. Simulation methods are superior to simply using point estimates for stochastic parameters because we can estimate statistics of interest such as expectations and probabilities based upon a large sample of solutions (referred to as "runs") for random values of λ. As we increase the number of runs, we increase "the confidence" in our estimates. That is, we either increase the confidence level that a fixed interval covers an estimate's true value or we decrease the width of the confidence interval for a fixed confidence level (Law and Kelton, 1991).

Reaching a desired level of confidence in our estimates can be burdensome because solving a single VI may be computationally expensive. Hence, our goal is to generate a sample space, N, large enough to reach a desired level of confidence for statistics of interest without resorting to solving all N parametric problems, thereby realizing substantial computational savings. To achieve this goal, we propose an iterative method where we solve or estimate a fixed number of PVI(F(x),λ^n) (n is an index) for each iteration. Following each iteration we may evaluate the sufficiency of our sample size, $N=\sum_{i=0}^{I} N_i$ based upon our desired level of confidence. To save computation time, we use the gradient information at two actual solutions for λ^n (e.g. λ^l and λ^r) to determine whether an estimate may be used in lieu of an actual solution for interim values of λ in the open interval, (λ^{l}, λ^{r}). The definitions of these estimates are detailed in this section and illustrated in Section 4.

In this section, we start with an explanation of how the methodology proceeds which includes a discussion of how we generate values for λ for our runs. Then, we formalize a summary of our method followed by some observations regarding the general shape of vector elements and a discussion of how our methodology might be modified in light of these observations.

Given a SVI(F(x),λ) where λ is an uncertain parameter with cumulative distribution G(λ), our objective is to generate a sample of solutions to PVI(F(x),λ^{n}) large enough to estimate statistics of interest. We solve or estimate solutions to PVI(F(x),λ^{n}) for specific values of λ. We select those values using a quasi-Monte Carlo method. The Monte Carlo method relies on a random number generator to generate values of λ with probabilities that, over the long run, approach their true probabilities of occurrence. The quasi-Monte Carlo method that we use is mathematically equivalent to the Monte Carlo method.

To generate a quasi-Monte Carlo sample of size N we proceed as follows. First, we select N values for u that are equally spaced between 0 and 1. Thus, u={0, 1/N-1, 2/N-1, … N-2/N-1, 1.0} is a set of such values. These may be thought of as numbers generated from the uniform distribution, U[0,1]. Next, we apply the Inverse Transform Method (Law and Kelton, 1991) to generate values for λ. For λ with distribution G(λ), we transform $\lambda = G^{-1}(u)$. Hence, $\lambda = \{G^{-1}(0), G^{-1}(1/N-1), G^{-1}(2/N-1), \ldots G^{-1}(N-2/N-1), 1.0\}$ or the set $\{\lambda = G^{-1}(n/N-1)$, n integer, $0 \leq n \leq N-1\}$ is our quasi-Monte Carlo sample.

For each iteration of our proposed methodology, we must ensure that our sample is quasi-Monte Carlo. For example, we may start with a sample size of as few as three, $N_0=3$. Hence, $\lambda = \{G^{-1}(0.0), G^{-1}(0.5), G^{-1}(1.0)\}$ would be our initial sample. To maintain a quasi-Monte Carlo sample for Iteration 1, I=1, we must solve or estimate $N_1=2$ solutions for $\lambda = \{G^{-1}(0.25), G^{-1}(0.75)\}$. And at the end of Iteration 1, our sample size is N=5. In general, starting with an initial sample size, k, we must solve or estimate $N_I=(k-1)2^{II-1}$ solutions to PVI(F(x),λ^n) for each iteration, I, and the sample size is $N=(k-1)2^I+1$ after I iterations. In practice, we may measure the trade off between increasing our sample size or increasing our confidence before beginning a new iteration.

We define e as the index of the PVI we wish to add to our sample space as we execute the main step to ultimately solve or estimate the solution for PVI(F(x),λ^e). *l* (r) is the index of λ^l (λ^r) for which the *actual* solution to (PVI(F(x),λ^l) ((PVI(F(x),λ^r)) has been calculated and λ^l (λ^r) is the closest such value to the left(right) of λ^e (see Figure 1).

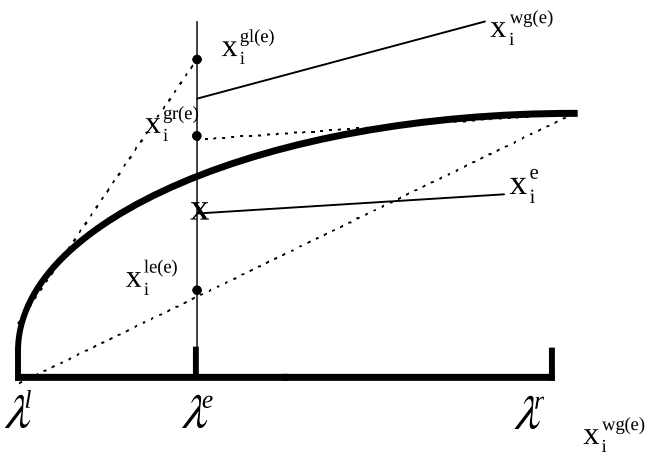

Figure 1: Illustration of a Convex Vector Element

An actual solution must be used for our estimates because we need the gradients to calculate them. $X^{le(e)}$, $X^{gl(e)}$, $X^{gr(e)}$ and $X^{wg(e)}$ are the linear estimate, gradient estimate from left actual solution, gradient estimate from right actual solution, and a weighted gradient estimate, respectively. X^e is either the actual solution to PVI(F(x),λ^e) or an estimate for its solution. We define these estimates as follows:

$$X^{le(e)} = \left|\frac{\lambda^e - \lambda^l}{\lambda^r - \lambda^l}\right| X^r + \left(\frac{\lambda^r - \lambda^e}{\lambda^r - \lambda^l}\right) X^l$$

$$X^{gl(e)} = X^l + \left(\lambda^e - \lambda^l\right)\frac{dX^l}{d\lambda}$$

$$X^{gr(e)} = X^r - \left(\lambda^r - \lambda^e\right)\frac{dX^r}{d\lambda}$$

$$X^{wg(e)} = \left|\frac{\lambda^r - \lambda^e}{\lambda^r - \lambda^l}\right| X^{gl(e)} + \left(\frac{\lambda^e - \lambda^l}{\lambda^r - \lambda^l}\right) X^{gr(e)}$$

$$X^e = \frac{X^{wg(e)} + X^{le(e)}}{2} \qquad (1)$$

We begin our proposed methodology by generating an initial sample of k actual solutions for k values of λ. Then, for each k-1 pairs of adjacent values of λ, λ^l and λ^r, we call the main step. The main step is called k-1 times for iteration, I=1 and, in general, it is called $(k-1)2^{I-1}$ times for iteration, I. We calculate two gradient-based estimates ($X^{gr(e)}$ and $X^{gl(e)}$) and a linear estimate ($X^{le(e)}$) for an interim value, λ^e. Under some assumptions of continuity, these estimates tell us the general shape of the curve for each vector element. For example, in Figure 1, the gradient estimates are both greater than the linear estimate indicating a generally concave shape. Conversely, if both gradient estimates are less than the linear estimate, a generally convex shape is indicated. Otherwise, there is an inflection. Since we will be comparing the weighted gradient estimate to the linear estimate to determine the appropriateness of estimating a solution for λ^e, we must test for an inflection. If there is an inflection, we calculate the actual solution, X^e, for λ^e by solving PVI(F(x),λ^e).

If there is not an inflection, we apply an acceptance criterion(see step 4 below) of comparing the absolute difference of the weighted gradient estimate and the linear estimate relative to the linear estimate for each vector element. If these two estimates are relatively close for all vector elements then (1) is a good estimate the solution of PVI(F(x),λ^e). If not, we obtain an actual solution by solving PVI(F(x),λ^e) and calculate its gradient. We complete steps 1-5 (see summary below) for each k-1 pairs to complete the first iteration, I=1. At the end of each iteration our sample size is $N=k+(k-1)2^I$. We determine whether our desired level of confidence is met before beginning a new iteration. After a sufficient number of iterations, we complete our methodology and calculate statistics of interest.

Before formalizing our proposed methodology, we introduce an index, $n\in[0,M]$, for u^n, λ^n, and our sample of solutions and estimates, X^n to PVI(F(x),λ^n). For an initial size sample of, $N_0=k$, we choose $M \in \{(k-1)2^m+1\}$ where m is an integer. M may be arbitrarily large and should be selected to be larger than the total N solutions we believe necessary to achieve our desired confidence level. Thus, $u^n=\{u^0, u^1, u^2,\ldots u^{M-2}, u^{M-1}\}=$ $\{0.0, 1/(M-1), 2/(M-1), \ldots (M-2)/(M-1), 1.0\}$or the set, $u^n =\{n/(M-1$, n integer, $0\leq n \leq M-1\}$. Our initial sample of size k for λ is the set, $\lambda^{k'}=\{G^{-1}(u^{k'})$, $k'=y(M-1)/(k-1)$, y integer, $0\leq y \leq k-1\}$. After solving PVI(F(x),$\lambda^{k'}$) for this same set, we execute the main step for the set $\lambda^e =\{ G^{-1}(u^e)$, $e= y(M-1)/2(k-1)$, y integer, $0\leq y \leq k-1\}$.

A summary of our method follows:

Initialization. The analyst specifies suitable values for the initial sample, k, the index $M \in \{(k-1)2^m+1;\ m\ \text{integer}\}$ and ε. ε is the maximum acceptable relative error for any single element x_i of the solution vector X^n to PVI(F(x),λ^n). Define U(0,1) to be the uniform distribution from 0 to 1 and u^n to be uniformly spaced points such that $u^n = 0$; $u^{M-1}=1$. Let $\lambda^n = G^{-1}(u^n)$. Solve PVI(F(x),$\lambda^{k'}$), for k'= y(M-1)/(k-1) for all y from 0 to k-1.

Let I=1, N=k.

Execute the main step for the set $\lambda^e = \{G^{-1}(u^e)$, e= y(M-1)/2(k-1), y integer,
$0 \le y \le k-1\}$.

Main Step

1. Calculate Linear Estimate: $X^{le(e)}$
2. Calculate Gradient Estimates: $X^{gl(e)}$ and $X^{gr(e)}$
3. If for any vector element, x_i, $x_i^{gr(e)} \le x_i^{le(e)} \le x_i^{gl(e)}$ OR $x_i^{gl(e)} \le x_i^{le(e)} \le x_i^{gr(e)}$ holds,
   Then there is an inflection;
   Calculate actual solution for $X^{e:}$ by solving PVI(F(x),λ^e)
   Go to 5.
   Else Calculate Weighted Gradient Estimate: $X^{wg(e)}$
4. If for any vector element, $x_{i,}$ $\left|\dfrac{x_i^{wg(e)} - x_i^{le(e)}}{x_i^{le(e)}}\right| > \varepsilon$
   Then estimation is not acceptable;
   Calculate actual solution for $X^{e:}$ by solving PVI(F(x),λ^e)
   Go to 5.
   Else Calculate Estimate for solution to PVI(F(x),λ^e) by using (1).

5. When X^e has been solved or estimated for all values of e, evaluate the sufficiency of the sample size $N=N+N_I$.
If N is sufficient
   END.

Else, I = I+1. Execute the main step for the set
$\lambda^e = \{ G^{-1}(u^e)$, e= y(M-1)/$2^I$(k-1), y integer, $0 \le y \le 2^{I-1}(k-1)\}$.

In addition to indicating the general shape of a vector element's curve, the estimates in Steps 1 and 2 also give us important information regarding the range of possible values for a solution, X^e, to PVI(F(x),λ^e). Specifically, if the estimates indicate a vector element's general shape is:

1. Concave, then $x_i^{le(e)} \le x_i^e \le \text{Min}\{x_i^{gr(e)}, x_i^{gl(e)}\}$
2. Convex, then $x_i^{le(e)} \ge x_i^e \ge \text{Max}\{x_i^{gr(e)}, x_i^{gl(e)}\}$ (2)
3. An Inflection, then $\text{Max}\{x_i^{gr(e)}, x_i^{gl(e)}\} \ge x_i^e \ge \text{Min}\{x_i^{gr(e)}, x_i^{gl(e)}\}$.

Our ability to specify boundaries on x_i presents the possibility of an alternative acceptance criterion for Steps 3 and 4. That is, we could calculate the size of the range of possible values for each vector element and compare that to a specified threshold.

We may use these bounds for possible values for x_i to further reduce the number of PVI that we must solve when we are only interested in estimating a probability. Specifically, for estimating $P(a< x_i<b)$, then sequence numbers 3 and 4 in the Main Step reduce to determining whether or not the bounds for x_i, as defined by (2), lie completely within or outside the range, (a,b). We include an example of this special case in Section 4.

In this section, we have presented a formal version of our method to generate a sample of N solutions for a SVI that is easy to code for computational purposes. The key contribution of this method is that we are using gradient information at actual solution points to estimate additional points to generate a quasi-Monte Carlo sample of solutions for a SVI. In the next section, we will demonstrate that our methodology significantly reduces the computational burden of generating a large sample of solutions by presenting an illustrative example.

4. Illustrative Example and Results

In this section, we present an adaptation of Harker's (1993) Spatial Price Equilibrium (SPE) problem to illustrate our methodology and to demonstrate that we can realize significant computational savings. We also include a brief example of how we may further reduce the number of PVI's that must be solved for the special case of calculating a single probability. After defining the SVI for this illustration, we provide detailed steps and calculations. Then we show results for two different distributions for λ. We compare our results to those obtained by simply using a the mean of the distribution, G(λ) as a point estimate for λ. We also compare our results to those obtained by solving all N=8193 PVIs. Using our methodology we only need to solve 36 PVIs with a threshold value, $\varepsilon = .01$. We close this section with confidence intervals for the expected value of X.

The problem is a SPE transportation model with two supply nodes and three demand nodes, as depicted in Figure 2.

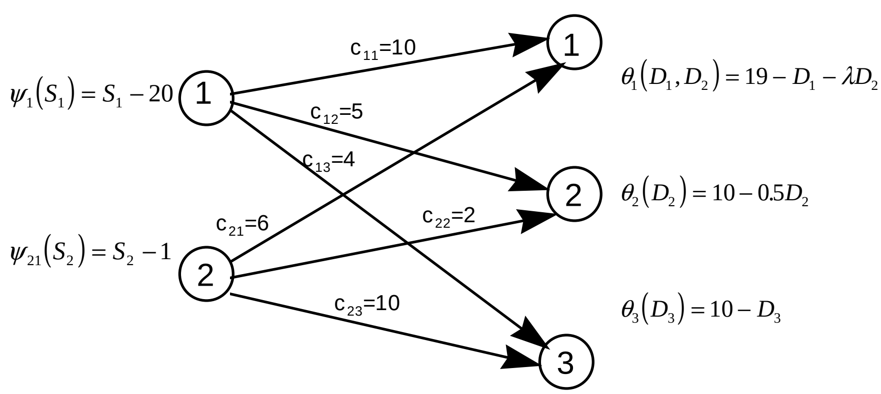

Figure 2: Model of Harker's SPE problem

The supply price, ψ_i, and demand price, θ_j, are functions of supply, S_i and demand D_j at nodes i and j, respectively. Exceptionally, at demand node 1 price is a function of demand at nodes 1 and 2. Importantly, note that the price function at demand node 1 includes a stochastic parameter, λ. As λ increases, the demand price at node 1 decreases for positive values of demand at node 2. The shipment costs, c_{ij}, from supply node i to demand node j are shown on the corresponding arrows. In vector form, $C=[c_{11}, c_{12}, c_{13}, c_{21}, c_{22}, c_{23}]^T$. The vector elements x_{ij} of the solution vector, $X= [x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}]^T$, are the supply amounts from i to j.

We make the following definitions:

$$\Psi(S)=\left[\psi_1(S_1),\psi_1(S_1),\psi_1(S_1),\psi_2(S_2),\psi_2(S_2),\psi_2(S_2)\right]$$

$$\theta(D)=\left[\theta_1(D_1,D_2),\theta_2(D_2),\theta_3(D_3),\theta_1(D_1,D_2),\theta_2(D_2),\theta_3(D_3)\right]$$

$$\text{where } S_i = \sum_{j=1}^{3} x_{ij} \text{ and } D_j = \sum_{i=1}^{2} x_{ij}.$$

Our goal is to simulate SVI(F(x), λ), F(x)=C+Ψ(S)-θ(D), λ~G(λ) where G(λ) is the cumulative distribution function for the random variable λ.

For illustrative purposes, we simulated the SVI for two distributions of λ. Moreover, we chose a sufficiently broad range for our distributions to demonstrate the methodology’s efficacy. Specifically, we ensured that we were not sampling over a solution space where the changes across all vector elements were trivially linear. The first distribution for λ, $G_1(\lambda)$, is uniformly distributed between 0 and 4, and the second distribution, $G_2(\lambda)$, is a triangle distribution on the same closed interval, [0,4] with a mode of 2. Triangle distributions are often used in the absence of data (Law and Kelton, 1991). The values for the interval and the mode are based upon expert opinion where the mode is deemed “most likely” and the interval is established by minimum and maximum possible values for the parameter of interest. Note that both distributions share a common mean of 2. Formally, our distributions are defined:

$$G_1(\lambda) = \begin{cases} 0 & \text{if } \lambda < 0 \\ \frac{\lambda}{4} & \text{if } 0 \le \lambda \le 4 \\ 1 & \text{if } \lambda > 4 \end{cases} \qquad G_2(\lambda) = \begin{cases} 0 & \text{if } \lambda < 0 \\ \frac{\lambda^2}{8} & \text{if } 0 \le \lambda \le 0 \\ 1 - \frac{(4-\lambda)^2}{8} & \text{if } 0 < \lambda \le 4 \\ 1 & \text{if } \lambda > 4 \end{cases}$$

To demonstrate our methodology we started with k=3 initial solutions and chose to set M=8193 (m=12). We solved or estimated all 8193 PVI's (N=8193) to emphasize that our method may be used to generate a large sample based upon a relatively few number of actual solutions. Moreover, our large value of N together with a small ε=.01 allowed us to calculate some reasonable confidence intervals for the expected value. These confidence intervals are presented at the end of this section. In the paragraphs and tables that follow we detail the main step for Iteration 1 and provide estimates or solutions for Iterations 1-3 in Table 2 for SVI(F(x), λ) with $\lambda \sim G_1(\lambda)$. Then we present our results for both distributions, $G_1(\lambda)$ and $G_2(\lambda)$ in Tables 3 and 4.

Initialization: ε=.01, N= k=3, M=8193, I=1
Let $\lambda^n = G^{-1}(u^n) = 4.0 * u^n$
Solve PVI(F(x),$\lambda^{k'}$) for k'=0, 4096, 8192 (Table 1)

Table 1: Initialization: Solutions and Gradients for PVI(F(x),$\lambda^{k'}$)

k'	$u^{(k')}$	$\lambda^{(k')}$	$X^{(k')}$	$dX^{(k')}/d\lambda$
0	0.0	0.0	[0.0,8.0,11.0,11.0,4.0,0.0]	[0.0,2.4,-1.2,10.8,-4.8,0.0]
4069	0.5	2.0	[1.142857,9.428571,9.714286, 26.428571,0.0,0.0]	[2.693878,-1.346939,-0.673469, 3.367347,0.0,0.0]
8192	1.0	4.0	[5.333333,7.333333,8.666667, 31.6667,0.0,0.0]	[1.629630,-0.814815,-0.407407, 2.037037,0.0,0.0]

Iteration 1. Main Step: e= {2048, 6144}

For e= 2048, $\lambda^{(e)}$=1.0

Calculate Linear Estimate:

$X^{le(2048)}$=[0.571429, 8.714286, 10.357143, 18.714286, 2.0. 0.0]^T

Calculate Gradient Estimates:

$X^{gl(2048)}$=[0.0, 10.4, 9.8, 21.8, -0.8, 0.0]^T

$X^{gr(2048)}$=[-1.551021, 10.775510, 10.387754, 23.061224, 0.0, 0.0]^T

Calculate $X^{wg(2048)}$=[-0.775511, 10.587755, 10.093877, 22.430612, -0.4, 0.0]^T

Since $\left|\frac{x_{11}^{wg(e)} - x_{11}^{le(e)}}{x_{11}^{le(e)}}\right| = 2.357 > \varepsilon$ Solve PVI(F(x),λ^{2048}) (Table 2)

(If one vector element fails to reach the threshold, we need not check others.)

For e= 6144, λ=3.0

Calculate Linear Estimate:

$X^{le(6144)}$=[3.238095, 8.380952, 9.190477, 29,047621, 0.0, 0.0]^T

Calculate Gradient Estimates:

$X^{gl(6144)}$=[3.836735, 8.081632, 9.040818, 29.795918, 0.0, 0.0]^T

$X^{gr(6144)}$=[3.703703, 8.148148, 9.074074, 29.0476205, 0.0, 0.0]^T

Calculate $X^{wg(6144)}$=[3.770219, 8.114890, 9.057446, 29.712776, 0.0, 0.0]^T

Since $\left|\frac{x_{11}^{wg(e)} - x_{11}^{le(e)}}{x_{11}^{le(e)}}\right| = 0.164 > \varepsilon$ Solve PVI(F(x),λ^{6144}) (Table 2)

N=N+2=5; I=I+1=2

Iteration 2. Main Step: e={1024, 3072, 5120, 7168} (See Table 2).

Table 2: Iterations 1-3: Estimates or Solutions for PVI(F(x),λ^e)

I	e	λ^e	Estimate or Solve	$X^{(e)}$	$dX^{(e)}/d\lambda$	N
1	2048	1.0	Solve	[0.0, 10.0, 10.0, 20.0, 0.0, 0.0]	[0.0, 0.0, 0.0, 8.333333, 0.0, 0.0]	
	6144	3.0	Solve	[3.5, 8.25, 9.125, 29.375, 0.0, 0.0]	[2.0625, -1.03125, -0.515625, 2.578125, 0.0, 0.0]	5
2	1024	0.5	Solve	[0.0,9.090909,10.454545, 15.909091, 1.818182, 0.0]	[0.0, 1.983471, -0.991736, 8.925620, -3.966942, 0.0]	
	3072	1.5	Solve	[0.0, 10.0, 10.0, 24.166667, 0.0, 0.0]	[0.0, 0.0, 0.0, 8.333333, 0.0, 0.0]	
	5120	2.5	Solve	[2.4, 8.8, 9.4, 28.0, 0.0, 0.0]	[2.346667, -1.173333, -0.586667, 2.933333, 0.0, 0.0]	
	7168	3.5	Estimate	[4.470775, 7.764612, 8.882306, 30.588470,0.0, 0.0]		9
3	512	0.25	Estimate	[0.0 8.571488, 10.714256, 13.571694, 2.857025, 0.0]		
	1536	0.75	Solve	[0.0, 9.565217, 10.217391, 18.043478, 0.869565, 0.0]	[0.0, 1.814745, -0.907372, 8.166352, -3.629490, 0.0]	
	2560	1.25	Estimate	[0.0, 10.0, 10.0, 22.083333, 0.0, 0.0]		
	3584	1.75	Solve	[0.444444, 9.777778, 9.888889, 25.555556, 0.0, 0.0]	[2.897119, -1.448560, -0.724280, 3.621399, 0.0, 0.0]	
	4608	2.25	Estimate	[1.793129, 9.103435, 9.551718, 27.241411, 0.0, 0.0]		
	5632	2.75	Estimate	[2.967760, 8.516120, 9.258060, 28.709701, 0.0, 0.0]		
	6656	3.25	Estimate	[3.998915, 8.000542, 9.000271, 29.998643, 0.0, 0.0]		
	7680	3.75	Estimate	[4.915581, 7.542209, 8.771105, 31.144477, 0.0, 0.0]		17

Table 3 summarizes our results when $\lambda \sim G_1(\lambda)$. For each of three threshold values, we simulated 8193 runs of SVI(F(x), λ). We calculated the expected value and the probability that a vector element is positive. For $\varepsilon = .01$, we solved 36 PVI's and obtained results nearly identical to those obtained by solving all 8193 PVI's. These results are substantially different than those obtained by simply substituting the mean as a point estimate. For example, the point estimate for x_{11} is positive, yet our methodology shows it has nearly a 40% chance of being zero. Our simulation results confirm this conclusion. Table 4 summarizes our results when $\lambda \sim G_2(\lambda)$.

Table 3: Expected Values and P(X>0) for SVI(F(x), λ), λ is Uniformly Distributed

Solution Method	Point Estimate	quasi-Monte Carlo Simulation	New Method ($\varepsilon = .01$)	New Method ($\varepsilon = .05$)	New Method ($\varepsilon = .10$)
Number of PVI(F(x), λ) solved	1	8193	36	28	27
$E(x_{11})$	1.142957	1.764999	1.765000	1.765026	1.765071
$E(x_{12})$	9.428571	8.882586	8.882586	8.882577	8.882555
$E(x_{13})$	9.714286	9.676206	9.676208	9.676200	9.676188
$E(x_{21})$	26.428571	24.524010	24.524019	24.524069	24.524120
$E(x_{22})$	0.000000	0.0469834	0.0469833	0.0469825	0.0469825
$E(x_{23})$	0.000000	0.000000	0.000000	0.000000	0.000000
$P(x_{11}>0)$	1	0.600024	0.600024	0.600024	0.600024
$P(x_{12}>0)$	1	1	1	1	1
$P(x_{13}>0)$	1	1	1	1	1
$P(x_{21}>0)$	1	1	1	1	1
$P(x_{22}>0)$	0	.249969	.249969	.249969	.249969
$P(x_{23}>0)$	0	0	0	0	0

Table 4: Expected Values and P(X>0) for SVI(F(x), λ), λ is Triangularly Distributed

Solution Method	Point Estimate	quasi-Monte Carlo Simulation	New Method ($\varepsilon = .01$)	New Method ($\varepsilon = .05$)	New Method ($\varepsilon = .10$)
Number of PVI(F(x), λ) solved	1	8193	36	29	27
$E(x_{11})$	1.142957	1.468445	1.468438	1.468404	1.468227
$E(x_{12})$	9.428571	9.189774	9.189782	9.189805	9.189921
$E(x_{13})$	9.714286	9.670892	9.670889	9.670893	9.670927
$E(x_{21})$	26.428571	25.418407	25.418390	25.418379	25.418262
$E(x_{22})$	0.000000	0.151996	0.151995	0.151983	.151927
$E(x_{23})$	0.000000	0.000000	0.000000	0.000000	0.000000
$P(x_{11}>0)$	1	0.679971	0.679971	0.679971	0.679971
$P(x_{12}>0)$	1	1	1	1	1
$P(x_{13}>0)$	1	1	1	1	1
$P(x_{21}>0)$	1	1	1	1	1
$P(x_{22}>0)$	0	.124985	.124985	.124985	.124985
$P(x_{23}>0)$	0	0	0	0	0

Using our proposed methodology to estimate or solve PVI's, we are able to calculate probabilities that are identical to our simulation results with as few as 27 solved PVIs. As discussed in Section 3, we could further reduce the number of PVIs that need to be solved if we are only interested in the special case of simply calculating a probability such as $P(a<x_{ij}<b)$ for a single vector element. For our example, this interest might arise if different distributors were responsible for each of our six routes. Then a representative distributor might only be interested in the probability of positive shipment or the probability x_{ij} takes on values over some range due to capacity constraints or other considerations. Under either of these scenarios, we can take advantage of the boundaries that we may establish for vector elements of X (Section 3).

For this special case, we alter our proposed methodology by examining the general shape of the curve and establishing upper and lower bounds on the vector element of interest. Then we determine if our range of interest is completely within or outside those boundaries for each value of λ. If the range of interest is not completely within or outside those boundaries, we must calculate the solution to its associated PVI. We present a brief example to calculate $P(x_{11}>0)$ for SVI(F(x), λ), $\lambda \sim G_1(\lambda)$.

Our initialization step remains the same as above with the exception that a threshold value is not necessary.

Initialization: N= k=3, M=8193, I=1
Let $\lambda^n = G^{-1}(u^n) = 4.0 * u^n$
Solve PVI(F(x),λ^n) for n=0, 4096, 8192 (Table 1.)

We choose N=M=8193 so that we may compare our results to those in Table 3. Table 5 summarizes the results of the first few iterations. *l*, r are the indices of the nearest values of λ to the left and right of λ for which actual solutions have been calculated. If the boundary information for x is inconclusive, then we must solve PVI(F(x), λ^e).

Table 5: Iterations 1 and 2: Bounds on x_{11}^e

						bounds	

e	l , r	λ^e	$x_{11}^{le(e)}$	$x_{11}^{gl(e)}$	$x_{11}^{gr(e)}$	lower, upper	Conclusion
	Iteration 1.						
2048	0,4096	1.0	0.5714	0.0000	-1.5510	0.0000, 0.5714	Solve PVI(F(x), λ^{2048})
6144	4096, 8193	3.0	3.2381	3.8367	3.7037	3.2381, 3.7037	$x_{11}^{6144} > 0$
	Iteration 2.						
1024	0, 2048	0.5	0.0000	0.0000	0.0000	0.0000, 0.0000	$x_{11}^{2048} = 0$
3072	2048, 4096	1.5	.5714	0.0000	-.2041	0.0000, 0.5714	Solve PVI(F(x), λ^{3072})
5120	4096, 8193	2.5	2.1905	2.4900	2.8894	2.1905, 2.4900	$x_{11}^{5120} > 0$
7168	4096, 8193	3.5	4.2857	5.1837	4.5185	4.2857, 4.5185	$x_{11}^{7168} > 0$

We continued to solve PVIs as necessary until we determined the status of all 8193 solutions. For this example, we only need to calculate a total of 15 PVI's to determine $P(x_{11}>0)$= 4916/8193=0.6000 which is the same as our result in Table 3.

Our results for calculating confidence intervals for E(X) using our methodology are also very encouraging. The estimated 90% confidence intervals (CI) for our new methodology are compared to the results obtained by solving all 8193 PVI for a quasi-Monte Carlo simulation in Table 6 for SVI(F(x), λ) with λ distributed triangularly. We estimated our 90% confidence intervals for the new methodology by using the mean and a sample variance of our sample of estimated and actual solutions for threshold, $\varepsilon = .01$.

Table 6: 90% Confidence Intervals (CI) for E(X) and Estimated 90% Confidence Intervals (CI) for Estimated E(X) for SVI(F(x), λ), λ~G_2(λ)

	quasi-Monte Carlo Simulation		New Methodology ($\varepsilon = .01$)
$E(x_{11})$ 90% CI	1.468445 (1.441731,1.495159)	Estimated $E(x_{11})$ 90% CI	1.468438 (1.468143,1.468733)
$E(x_{12})$ 90% CI	9.189774 (9.189633,9.189915)	Estimated $E(x_{12})$ 90% CI	9.189782 (9.189641,9.189923)
$E(x_{13})$ 90% CI	9.670892 (9.670807,9.670977)	Estimated $E(x_{13})$ 90% CI	9.670889 (9.670804,9.670974)
$E(x_{21})$ 90% CI	25.418407 (25.417604,25.419210)	Estimated $E(x_{21})$ 90% CI	25.418390 (25.417587,25.419193)
$E(x_{22})$ 90% CI	0.151996 (0.151893,0.152099)	Estimated $E(x_{22})$ 90% CI	0.151995 (0.151892,0.152098)
$E(x_{23})$ 90% CI	0.000000 (0.000000,0.000000)	Estimated $E(x_{23})$ 90% CI	0.000000 (0.000000,0.000000)

5. Conclusions and Further Research

In this paper, we have made a major contribution to the study of general equilibrium by presenting a method for analyzing the SVI that uses a quasi-Monte-Carlo simulation technique and relies on gradient information at solutions to estimate rather than calculate most of the N solutions to its associated PVI. Our illustrative example demonstrates that we need only solve 36 versus 8193 PVI to estimate expectations with reasonable confidence intervals. For the special case of estimating a single probability we reduced the number of PVIs that need to be solved to just 15. This represents a significant computational saving and relieves the burdensome task of calculating large numbers of PVI's. This task has, heretofore, discouraged analysts from undertaking a comprehensive analysis of SVI. We hope that this work will encourage practitioners to abandon the current practice of using point estimates. And we look forward to continued research in this area.

While the analysis of a single stochastic parameter is an important first step in the study of SVI, we recognize the necessity of extending our method to multiple parameters. This is an important area of further research. Other important questions may, at this point, be matters of art versus science. For example, the selection of N and epsilon will be situationally dependent. We have presented a SPE as an illustration of our methodology. We look forward to further establishing the efficacy of our methodology by applying it to a broad class of equilibrium problems.

Notes

1. When actual solutions are calculated, we use the PATH solver in GAMS to solve PVI(F(x), λ^n) throughout this paper.

Acknowledgments

This work was supported by the National Science Foundation under grants SBR 96-02053 and DMI-9634808.

References

Baiocchi, C. and A. Capelo, *Variational and Quasivariational Inequalities: Applications to Free-Boundary Problems* (John Wiley, New York, 1984).

Cottle, R.W., F. Giannessi and J.L. Lions, eds., *Variational Inequalities and Complementarity Problems: Theory and Applications* (John Wiley, New York, 1980).

De Wolf, D and Y. Smeers, "A Stochastic Version of a Stackelberg-Nash-Cournot Equilibrium Model," *Management Science*, 43 (Feb 1997), 190-197.

Ferris, M.C. and K. Sinapiromsaran, *Complementarity Problem Net*, http://www.cs.wisc.edu/cpnet/, Last modified, 8 October 1999.

Ferris, M.C. and T.S. Munson, "Interfaces to PATH 3.0: Design, Implementation and Usage," *Computational and Applied Optimization*, 12 (1999), 207-227.

Gurkan, G., A.Y. Ozge, and S.M. Robinson, "Sample-Path Solution of Stochastic Variational Inequalities," *Mathematical Programming*, 84 (Feb 1999), 313-333.

Harker, P.T., *Lectures on the Computation of Equilibria* (Center for Operations Research and Econometrics, Universite Catholique de Louvain, Belgium, 1993).

Harker, P.T. and J. S. Pang, "Finite-dimensional Variational Inequality and Nonlinear Complementarity Problems: A Survey of Theory, Algorighms and Applications," *Mathematical Programming* 48B (1990), 339-357.

Harrison, G. W., R. Jones, L.J. Kimbell, and R. Wigle, "How Robust Is Applied General Equilibrium Analysis?" *Journal of Policy Modelling* 15(1993), 99-115.

Harrison, G. W. and H.D. Vinod, "The Sensitivity Analysis of Applied General Equilibrium Models: Completely Randomized Factorial Sampling Designs," *The Review of Economics and Statistics*, 74,2(May 1992),357-362.

Kinderlehrer, D. and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications,* (Academic Press, New York, 1980).

Law, A. M. and W. D. Kelton, *Simulation Modeling and Analysis* (McGraw Hill, New York, 1991).

Mordukhovich, B. "Stability Theory for Parametric Generalized Equations and Variational Inequalities Via Nonsmooth Analysis," *Transactions of the American Mathematical Society*, 343 (1994), 609-657.

Nagurney, *A. Network Economics: A Variational Inequality Approach*, Revised 2d Ed, (Kluwer Academic Publishers, Boston, 1999).

Tobin, R.L. "Sensitivity Analysis for Variational Inequalities." *Journal of Optimization Theory and Applications* 48 (1986), 191-204.